V. Conclusion

The purpose of this paper is to propose an observer for nonlinear systems. This is done by modifying the well-known extended Kalman filter [5] for an effective treatment of the nonlinearities. We proved that under certain conditions this observer is an exponential observer [8] by choosing an appropriate Lyapunov function. For an alternative approach to the proposed observer we consider the nonlinearities as uncertainties and get a stabilization problem which is typical in robust control theory [9], [20]. This stabilization problem can be solved by H_∞-filtering techniques [2], [6], [10], [14] leading to the proposed modification of the extended Kalman filter. Using the proposed observer to estimate the rotor flux and the angular velocity of an induction motor we show that it can be easily applied to complex highly nonlinear systems. Numerical simulations show a good performance and an increased domain of convergence in comparison to the extended Kalman filter, i.e., we can tolerate larger initial estimation errors. Similar to H_∞-filtering problems, we have in general not a bounded positive definite solution of the corresponding Riccati differential equation, which does exist for the extended Kalman filter if the system satisfies certain observability conditions [1]. Therefore, a careful choice of the constants in this Riccati differential equation is of particular interest.

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Fractional-Order Systems and $PI^\lambda D^\mu$-Controllers

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***Abstract*— Dynamic systems of an arbitrary real order (fractional-order systems) are considered. A concept of a fractional-order $PI^\lambda D^\mu$-controller, involving fractional-order integrator and fractional-order differentiator, is proposed. The Laplace transform formula for a new function of the Mittag–Leffler-type made it possible to obtain explicit analytical expressions for the unit-step and unit-impulse response of a linear fractional-order system with fractional-order controller both for the open and closed loop. An example demonstrating the use of the obtained formulas and the advantages of the proposed $PI^\lambda D^\mu$-controllers is given.**

***Index Terms*— Fractional-order controllers, fractional-order systems, fractional differential equations, Laplace transforms, transfer functions.**

I. Introduction

Recently, several authors have considered mechanical systems described by fractional-order state equations [4], [5], [15], which means equations involving so-called fractional derivatives and integrals (for the introduction to this theory see [21]).

There are also several recent applications in electricity.

Le Méhauté and Crepy [14] have proposed a concept of a *fractance*—a new electrical circuit element, which has intermediate properties between resistance and capacitance. Such a device has been experimentally studied, for example, by Nakagawa and Sorimachi [18]. A circuit proposed by Oldham and Zoski [22] provides another example of a fractance.

A new capacitor theory developed by Westerlund [28] is based on the use of fractional derivatives.

New fractional derivative-based models are more adequate than the previously used integer-order models. This has been demonstrated, for instance, by Caputo [7], Nonnenmacher and Glöckle [19], Friedrich

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[12] and Westerlund [28]. Important fundamental physical considerations in favor of the use of fractional-derivative-based models were given by Caputo and Mainardi [8] and by Westerlund [27]. Fractional-order derivatives and integrals provide a powerful instrument for the description of memory and heredetary effects in various substances, as well as for modeling dynamical processes in fractal (as defined by Mandelbrote [16]) media. This is the most significant advantage of the fractional-order models in comparison with integer-order models, in which, in fact, such effects or geometry are neglected.

However, because of the absence of appropriate mathematical methods, fractional-order dynamic systems were studied only marginally in the theory and practice of control systems. Works by Bagley and Calico [4], Makroglou *et al.* [15], Axtell and Bise [3], Kaloyanov and Dimitrova [13], and Oustaloup [23] must be mentioned, but the study in the time domain has been almost avoided.

In this paper some effective and easy-to-use tools for the time-domain analysis of fractional-order systems are presented. A concept of a $PI^{\lambda}D^{\mu}$-controller, involving fractional-order integrator and fractional-order differentiator, is introduced. An example is provided to demonstrate the necessity of such controllers for the more efficient control of fractional-order systems. Such questions as uncertainties, noised input, and robustness are not the subject of this paper.

The idea of using fractional-order controllers for the control of dynamic systems belongs to Oustaloup, who developed the so-called *Commande Robuste d'Ordre Non Entier* (CRONE) controller, which is described in his book [24] along with examples of applications in various fields (see also other references in [24]). Oustaluop demonstrated the advantage of the CRONE controller in comparison with the PID-controller. The $PI^{\lambda}D^{\mu}$-controller, suggested in this paper, is a new type of a fractional-order controller, which also shows better performance when used for the control of fractional-order systems than the classical PID-controller.

II. Fractional-Order Systems

Contrary to the traditional approach, we will consider *the transfer functions of an arbitrary real order*. We call such systems the fractional-order systems. They include, in particular, traditional integer-order systems. It is important to realize that the words "fractional-order system" mean just "systems which are better described by fractional-order mathematical models."

Let us consider the fractional-order transfer function (FOTF) given by the following expression:

$$G_n(s) = \frac{1}{a_n s^{\beta_n} + a_{n-1} s^{\beta_{n-1}} + \cdots + a_1 s^{\beta_1} + a_0 s^{\beta_0}} \qquad (1)$$

where $\beta_k,\ (k = 0, 1, \cdots, n)$ is an arbitrary real number

$$\beta_n > \beta_{n-1} > \cdots > \beta_1 > \beta_0 > 0, \qquad a_k, (k = 0, 1, \cdots, n)$$

is an arbitrary constant.

In the time domain, the FOTF (1) corresponds to the n-term inhomogeneous fractional-order differential equation (FDE)

$$\begin{aligned} &a_n D^{\beta_n} y(t) + a_{n-1} D^{\beta_{n-1}} y(t) + \cdots \\ &\quad + a_1 D^{\beta_1} y(t) + a_0 D^{\beta_0} y(t) = u(t) \end{aligned} \qquad (2)$$

where $D^{\gamma} \equiv {}_0D_t^{\gamma}$ is Caputo's fractional derivative of order γ with respect to variable t and with the starting point at $t = 0$ [6], [7]

$$\begin{aligned} {}_0D_t^{\gamma} y(t) &= \frac{1}{\Gamma(1-\delta)} \int_0^t \frac{y^{(m+1)}(\tau)\, d\tau}{(t-\tau)^{\delta}}, \\ &\qquad (\gamma = m + \delta,\ m \in Z,\ 0 < \delta \le 1) \end{aligned} \qquad (3)$$

($\Gamma(z)$ is Euler's gamma function [1]).

If $\gamma < 0$, then one has a fractional integral of order $-\gamma$

$${}_0I_t^{-\gamma} y(t) = {}_0D_t^{\gamma} y(t) = \frac{1}{\Gamma(-\gamma)} \int_0^t \frac{y(\tau)\, d\tau}{(t-\tau)^{1+\gamma}}, \quad (\gamma < 0). \qquad (4)$$

The Laplace transform of the fractional derivative defined by (3) is [6], [7]

$$\int_0^{\infty} e^{-st} D^{\gamma} y(t)\, dt = s^{\gamma} Y(s) - \sum_{k=0}^{m} s^{\gamma-k-1} y^{(k)}(0). \qquad (5)$$

For $\gamma < 0$ (i.e., for the case of a fractional integral) the sum in the right-hand side must be omitted.

It is worth mentioning here that from the pure mathematical point of view there are several ways to interpolate between integer-order multiple integrals and derivatives. The most widely known and precisely studied is the Riemann–Liouville definition of fractional derivatives (see, e.g., [21], [26], and [17]). The main advantage of Caputo's definition in comparison with the Riemann–Liouville definition is that it allows consideration of easily interpreted conventional initial conditions such as $y(0) = y_0$, $y'(0) = y_1$, etc. Moreover, Caputo's derivative of a constant is bounded (namely, equal to zero), while the Riemann–Liouville derivative of a constant is unbounded at $t = 0$. The only exception is if one takes $t = -\infty$ as the starting point (lower limit) in the Riemann–Liouville definition. In such a case, the Riemann–Liouville fractional derivative of a constant is also zero, and this was used by Ochmann and Makarov [20]. However, one interested in transient processes could not accept placement of the starting point in $-\infty$, and in such cases Caputo's definition seems to be the most appropriate among others.

Formula (5) is a particular case of a more general formula given by Podlubny [25] for the Laplace transform of a so-called sequential fractional derivative introduced in [17].

To find the unit-impulse and unit-step response of the fractional-order system described by FDE (2), we need to evaluate the inverse Laplace transform of the function $G_n(s)$.

The problem of the Laplace inversion of (1), however, can appear in any field of applied mathematics, physics, engineering, etc., where the Laplace transform method is used. This fact along with the absence of the necessary inversion formula in tables and handbooks on the Laplace transform motivated us to give the general solution to this problem in two following sections.

III. New Function of the Mittag-Leffler-Type

The so-called Mittag–Leffler function in two parameters $E_{\alpha,\beta}(z)$ was introduced by Agarwal [2]. His definition was later modified by the authors of [11] to be

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)} \quad (\alpha > 0, \quad \beta > 0) \qquad (6)$$

Its kth derivative is given by

$$E_{\alpha,\beta}^{(k)}(z) = \sum_{j=0}^{\infty} \frac{(j+k)!\, z^j}{j!\, \Gamma(\alpha j + \alpha k + \beta)} \quad (k = 0, 1, 2, \cdots). \qquad (7)$$

We find it convenient to introduce the function

$$\mathcal{E}_k(t, y; \alpha, \beta) = t^{\alpha k + \beta - 1} E_{\alpha,\beta}^{(k)}(y t^{\alpha}) \quad (k = 0, 1, 2, \cdots). \qquad (8)$$

Its Laplace transform was (in other notation) evaluated by Podlubny [25]

$$\int_0^{\infty} e^{-st} \mathcal{E}_k(t, \pm y; \alpha, \beta)\, dt = \frac{k!\, s^{\alpha-\beta}}{(s^{\alpha} \mp y)^{k+1}} \qquad (9)$$

$$(\mathrm{Re}(s) > |y|^{1/\alpha}).$$

Another convenient property of $\mathcal{E}_k(t, y; \alpha, \beta)$, which we use in this paper, is its simple fractional differentiation [25]

$$ {}_0D_t^{\lambda} \mathcal{E}_k(t, y; \alpha, \beta) = \mathcal{E}_k(t, y; \alpha, \beta - \lambda) \quad (\lambda < \beta). \tag{10}$$

Other properties of function $\mathcal{E}_k(t, y; \alpha, \beta)$, such as special cases, asymptotic behavior, etc., can be obtained from (6)–(8) and the known properties [11] of the Mittag–Leffler function $E_{\alpha,\beta}(z)$.

IV. General Formula

Relationship (9) allows us to evaluate the inverse Laplace transform of (1) as follows. Let $\beta_n > \beta_{n-1} > \cdots > \beta_1 > \beta_0 > 0$. Then

$$\begin{aligned} G_n(s) &= \frac{1}{a_n s^{\beta_n} + a_{n-1}s^{\beta_{n-1}}} \frac{1}{1 + \frac{\sum_{k=0}^{n-2} a_k s^{\beta_k}}{a_n s^{\beta_n} + a_{n-1} s^{\beta_{n-1}}}} \\ &= \frac{a_n^{-1} s^{-\beta_{n-1}}}{s^{\beta_n - \beta_{n-1}} + \frac{a_{n-1}}{a_n}} \frac{1}{1 + \frac{a_n^{-1} s^{-\beta_{n-1}} \sum_{k=0}^{n-2} a_k s^{\beta_k}}{s^{\beta_n-\beta_{n-1}} + \frac{a_{n-1}}{a_n}}} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m a_n^{-1} s^{-\beta_{n-1}}}{\left(s^{\beta_n-\beta_{n-1}} + \frac{a_{n-1}}{a_n}\right)^{m+1}} \left(\sum_{k=0}^{n-2} \left(\frac{a_k}{a_n}\right) s^{\beta_k - \beta_{n-1}} \right)^m \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m a_n^{-1} s^{-\beta_{n-1}}}{\left(s^{\beta_n-\beta_{n-1}} + \frac{a_{n-1}}{a_n}\right)^{m+1}} \\ &\quad \times \sum_{\substack{k_0+k_1+\cdots+k_{n-2}=m \\ k_0\ge 0;\cdots,k_{n-2}\ge 0}} (m; k_0, k_1, \cdots, k_{n-2}) \\ &\quad \times \prod_{i=0}^{n-2} \left(\frac{a_i}{a_n}\right)^{k_i} s^{(\beta_i - \beta_{n-1})k_i} \\ &= \frac{1}{a_n} \sum_{m=0}^{\infty} (-1)^m \sum_{\substack{k_0+k_1+\cdots+k_{n-2}=m \\ k_0\ge 0;\cdots,k_{n-2}\ge 0}} (m; k_0, k_1, \cdots, k_{n-2}) \\ &\quad \times \prod_{i=0}^{n-2} \left(\frac{a_i}{a_n}\right)^{k_i} \frac{s^{-\beta_{n-1} + \sum_{i=0}^{n-2}(\beta_i - \beta_{n-1})k_i}}{\left(s^{\beta_n-\beta_{n-1}} + \frac{a_{n-1}}{a_n}\right)^{m+1}} \end{aligned} \tag{11}$$

where $(m; k_0, k_1, \cdots, k_{n-2})$ are the multinomial coefficients [1].

The term-by-term inversion, based on the general expansion theorem for the Laplace transform given in [9, § 22], using (9) gives the final expression for the inverse Laplace transform of function $G_n(s)$

$$\begin{aligned} g_n(t) &= \frac{1}{a_n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{\substack{k_0+k_1+\cdots+k_{n-2}=m \\ k_0\ge 0;\cdots,k_{n-2}\ge 0}} (m; k_0, k_1, \cdots, k_{n-2}) \\ &\quad \times \prod_{i=0}^{n-2} \left(\frac{a_i}{a_n}\right)^{k_i} \mathcal{E}_m \Bigg(t, -\frac{a_{n-1}}{a_n}; \beta_n - \beta_{n-1}, \beta_n \\ &\qquad + \sum_{j=0}^{n-2} (\beta_{n-1} - \beta_j) k_j \Bigg). \end{aligned} \tag{12}$$

Further inverse Laplace transforms can be obtained by combining (10) and (12). For instance, let us take

$$F(s) = \sum_{i=1}^{N} b_i s^{\alpha_i} G_n(s) \tag{13}$$

where $\alpha_i < \beta_n$, $(i = 1, 2, \cdots, N)$. Then the inverse Laplace transform of $F(s)$ is

$$f(t) = \sum_{i=1}^{N} b_i D^{\alpha_i} g_n(t) \tag{14}$$

where the fractional derivatives of $g_n(t)$ are evaluated with the help of (10).

V. The Unit-Impulse and Unit-Step Responses

The unit-impulse response of the fractional-order system with the transfer function (1) is given by formula (12), i.e.,

$$y_{\text{impulse}}(t) = g_n(t).$$

To find the unit-step response $y_{\text{step}}(t)$, one has to integrate (12) with the help of (10). The result is

$$\begin{aligned} y_{\text{step}}(t) &= \frac{1}{a_n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{\substack{k_0+k_1+\cdots+k_{n-2}=m \\ k_0\ge 0;\cdots,k_{n-2}\ge 0}} (m; k_0, k_1, \cdots, k_{n-2}) \\ &\quad \times \prod_{i=0}^{n-2} \left(\frac{a_i}{a_n}\right)^{k_i} \mathcal{E}_m \Bigg(t, -\frac{a_{n-1}}{a_n}; \beta_n - \beta_{n-1}, \beta_n \\ &\qquad + \sum_{j=0}^{n-2} (\beta_{n-1} - \beta_j) k_j + 1 \Bigg). \end{aligned} \tag{15}$$

VI. $PI^{\lambda}D^{\mu}$-Controller

As it will be shown in an example below, a good way to the more efficient control of fractional-order systems is to use fractional-order controllers. We propose a generalization of the PID-controller, which can be called the $PI^{\lambda}D^{\mu}$-controller because of involving an integrator of order λ and a differentiator of order μ. The transfer function of such a controller has the form

$$G_c(s) = \frac{U(s)}{E(s)} = K_P + K_I s^{-\lambda} + K_D s^{\mu} \quad (\lambda, \mu > 0) \tag{16}$$

where $G_c(S)$ is the transfer of the controller, $E(s)$ is an error, and $U(s)$ is controller's output.

The equation for the $PI^{\lambda}D^{\mu}$-controller's output in the time domain is

$$u(t) = K_P e(t) + K_I D^{-\lambda} e(t) + K_D D^{\mu} e(t). \tag{17}$$

Taking $\lambda = 1$ and $\mu = 1$, we obtain a classic PID-controller. $\lambda = 1$ and $\mu = 0$ give a PI-controller. $\lambda = 0$ and $\mu = 1$ give a PD-controller. $\lambda = 0$ and $\mu = 0$ give a gain.

All these classical types of PID-controllers are the particular cases of the fractional $PI^{\lambda}D^{\mu}$-controller (16). However, the $PI^{\lambda}D^{\mu}$-controller is more flexible and gives an opportunity to better adjust the dynamical properties of a fractional-order control system.

VII. Responses

Let us consider the open loop with the $PI^{\lambda}D^{\mu}$-controller (16) and the fractional-order controlled system with the transfer function $G_n(s)$ given by expression (1).

In the time domain, this open-loop system is described by the fractional-order differential equation

$$\sum_{k=0}^{n} a_k D^{\beta_k} y(t) = K_P w(t) + K_I D^{-\lambda} w(t) + K_D D^{\mu} w(t) \tag{18}$$

(here and below $w(t)$ means an input and $y(t)$ is the system output).

The transfer function of the considered open-loop system is

$$G_{\text{open}}(s) = (K_P + K_I s^{-\lambda} + K_D s^{\mu}) G_n(s). \tag{19}$$

Since (19) has the same structure as (13), the inverse Laplace transform for $G_{\text{open}}(s)$ can be found with the help of formula (14). Therefore, the unit-impulse response of the considered fractional-order open-loop system is

$$g_{\text{open}}(t) = K_P g_n(t) + K_I D^{-\lambda} g_n(t) + K_D D^{\mu} g_n(t) \tag{20}$$

where $g_n(t)$ is given by (12).

To find the unit-step response, one should integrate (20) using formula (10).

To obtain the unit-impulse and unit-step responses for a closed-loop unity-feedback control system with the $PI^{\lambda}D^{\mu}$-controller and the fractional-order controlled system with the transfer function $G_n(s)$ given by expression (1), one needs, at first, to replace $w(t)$ with $e(t) = w(t) - y(t)$ in (18). This step results in

$$\begin{aligned}\sum_{k=0}^{n} a_k D^{\beta_k} y(t) + K_P y(t) + K_I D^{-\lambda} y(t) + K_D D^{\mu} y(t) \\ = K_P w(t) + K_I D^{-\lambda} w(t) + K_D D^{\mu} w(t). \qquad (21)\end{aligned}$$

From (21) one obtains the following expression for the transfer function of the considered closed-loop system:

$$G_{\mathrm{closed}}(s) = \frac{K_P s^{\lambda} + K_I + k_D s^{\mu+\lambda}}{\sum_{k=0}^{n} a_k s^{\beta_k+\lambda} + K_P s^{\lambda} + K_I + K_D s^{\mu+\lambda}}. \qquad (22)$$

The unit-impulse response $g_{\mathrm{closed}}(t)$ is then obtained by the Laplace inversion of (22), which could be performed by rearranging in decreasing order of differentiation the addends in the denominator of (22) and applying after that relationships (12) and (14). To find the unit-step response, one should integrate obtained unit-impulse response with the help of (10).

VIII. Example

In this section we give an example showing the usefulness of the $PI^{\lambda}D^{\mu}$-controllers in comparison with conventional PID-controllers. We consider a fractional-order system, which *plays the role* of "reality," and its integer-order approximation, which plays the role of a "model." We underline that, at the first look, the model, obtained in a usual manner fits the data obtained from the "reality" well.

However, the PD-controller, designed on the base of the model, is shown to be not so suitable for the control of the "reality" as one can expect.

A good way to the improvement of the control is to use a controller of the similar "nature" as the "reality," i.e., a fractional-order PD^{μ}-controller. At this stage we assume that the fractional-order transfer function has been identified exactly.

It is important to realize that often, in fact, a structure of the model is postulated (in our example, the second-order differential equation model) and then the parameters of the model (in our case, the coefficients of the differential equation) are determined to provide suitable fitting of data obtained from the real object. However, as mentioned in the introduction, there are real systems which are better described by fractional-order differential equations. For such systems classical integer-order models, even of high order, will give less adequate results than fractional-order models. From this point of view, the example demonstrates some of the possible effects arising from the difference of the nature of the "reality" and the "model." It also indicates the necessity of development of methods for identification of parameters of fractional-order models, including the most appropriate *order of the model* (not the order of the real object).

A. Fractional-Order Controlled System

Let us consider a fractional-order controlled system, which *plays the role of "reality,"* with the transfer function

$$G(s) = \frac{1}{a_2 s^{\beta_2} + a_1 s^{\beta_1} + a_0} \qquad (23)$$

where we take $a_2 = 0.8$, $a_1 = 0.5$, $a_0 = 1$, $\beta_2 = 2.2$, $\beta_1 = 0.9$.

The fractional-order transfer function (23) corresponds in the time domain to the three-term inhomogeneous fractional-order differential equation

$$a_2 y^{(\beta_2)}(t) + a_1 y^{(\beta_1)}(t) + a_0 y(t) = u(t) \qquad (24)$$

with zero initial conditions $y(0) = 0$, $y'(0) = 0$, $y''(0) = 0$. (For simplicity, here and below we denote $y^{(\gamma)}(t) = {}_0D_t^{\gamma} y(t)$.)

The unit-step response is found by (15)

$$y(t) = \frac{1}{a_2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{a_0}{a_2}\right)^k \mathcal{E}_k\left(t, -\frac{a_1}{a_2}; \beta_2 - \beta_1, \beta_2 + \beta_1 k + 1\right). \qquad (25)$$

The introduced system "provides" data for obtaining its model.

B. Integer-Order Approximation

For the comparison purpose, let us approximate the considered fractional-order system by a second-order system. (Noticing that $\beta_2 = 2.2$ and $\beta_1 = 0.9$ are close to 2 and 1, respectively, one may expect good approximation. This remark, however, relates only to this model example.) Using the least squares method for the determination of coefficients of the resulting equation, we obtained the following approximating equation corresponding to (24):

$$\tilde{a}_2 y''(t) + \tilde{a}_1 y'(t) + \tilde{a}_0 y(t) = u(t) \qquad (26)$$

with $\tilde{a}_2 = 0.7414$, $\tilde{a}_1 = 0.2313$, $\tilde{a}_0 = 1$.

C. Integer-Order PD-Controller

Since the comparison of the unit-step responses shows good agreement, one may try to control the original system (24) by a controller designed for its approximation (26). This approach is, in fact, frequently used in practice, when one controls the real object by a controller designed for the model of that object.

The PD-controller with the transfer function

$$\tilde{G}_c(s) = \tilde{K} + \tilde{T}_d s \qquad (27)$$

was designed so that a unit step signal at the input of the closed-loop system with a unity feedback will induce at the output an oscillatory unit-step response with stability measure $St = 2$ (this is equivivalent to the requirement that the system must settle within 5% of the unit step at the input in 2 s: $T_s \leq 2s$) and damping ratio $\xi = 0.4$. In such a case, the coefficients for (27) take on the values $\tilde{K} = 20.5$ and $\tilde{T}_d = 2.7343$. (The plot of the unit-step response of the integer-order "model" controlled by the designed PD-controller has almost the same shape as the thin line in Fig. 2.)

For comparison purposes, we also computed the integral of the absolute error (IAE)

$$I(t) = \int_0^t |e(t)|\, dt$$

for $t = 5s$: $I(5) = 0.8522$.

Let us now apply this controller, designed for the optimal control of the approximating integer-order system (26), to the control of the approximated fractional-order system (24).

The differential equation of the closed loop with the fractional-order system defined by (23) and the integer-order controller defined by (27) has the following form:

$$\begin{aligned}a_2 y^{(\beta_2)}(t) + \tilde{T}_d y'(t) + a_1 y^{(\beta_1)}(t) + (a_0 + \tilde{K}) y(t) \\ = \tilde{K} w(t) + \tilde{T}_d w'(t). \qquad (28)\end{aligned}$$

This is the four-term inhomogeneous fractional differential equation, and the unit-step response of this system is found with the help

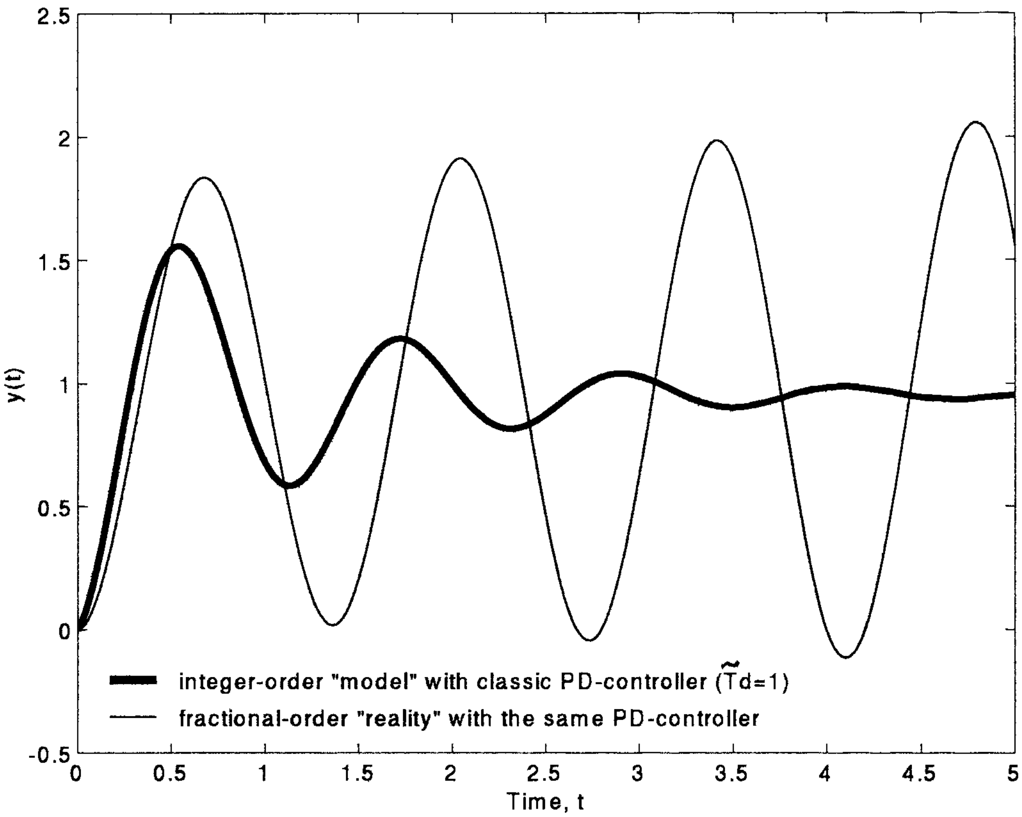

Fig. 1. Comparison of the unit-step response of the closed-loop integer-order system (thick line) and the closed-loop fractional-order system (thin line) with the same integer-order controller, optimally designed for the approximating system, for $\tilde{T}_d = 1$.

of (15), (14), and (10)

$$y(t) = \frac{1}{a_2} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{a_0 + \tilde{K}}{a_2} \right)^m \sum_{k=0}^{m} \binom{m}{k} \left(\frac{a_1}{a_0 + \tilde{K}} \right)^k \times \left\{ \tilde{K} \mathcal{E}_m \left(t, -\frac{\tilde{T}_d}{a_2}; \beta_2 - 1, \beta_2 + m - \beta_1 k + 1 \right) + \tilde{T}_d \mathcal{E}_m \left(t, -\frac{\tilde{T}_d}{a_2}; \beta_2 - 1, \beta_2 + m - \beta_1 k \right) \right\}. \quad (29)$$

The comparison of the perfomance shows that that the dynamic properties of the closed loop with the fractional-order controlled system and the integer-order controller, which was designed for the integer-order approximation of the fractional-order system, are considerably worse than the dynamic properties of the closed loop with the approximating integer-order system. The system stabilizes slower and has larger surplus oscillations. Computations show that, in comparison with the integer-order "model," in this case the IAE within 5 s time interval is larger by 76%. Moreover, the closed loop with the fractional-order controlled system is more sensitive to changes in controller parameters. For example, at the change of $\tilde{T}_d$ to value one, the closed loop with the fractional-order system (the "reality") is already unstable, whereas the closed loop with the approximating integer-order system (the "model") still shows stability (Fig. 1).

D. Fractional-Order Controller

We see that disregarding the fractional order of the original system (24), replacing it with the approximating integer-order system (26), and the application of the controller, designed for the approximating system, to the control of the original fractional-order system is not generally adequate.

An alternative and more successful approach in our example is to use the fractional-order PD^μ-controller characterized by the fractional-order transfer function

$$G_c(s) = K + T_d s^\mu. \quad (30)$$

Let us take $\beta_1 < \mu < \beta_2$. The differential equation of the closed-loop control system with the fractional-order system transfer (23) and the fractional-order controller transfer (30) can be written in the form

$$a_2 y^{(\beta_2)}(t) + T_d y^{(\mu)}(t) + a_1 y^{(\beta_1)}(t) + (a_0 + K) y(t) = K w(t) + T_d w^{(\mu)}(t). \quad (31)$$

We are interested in the unit-step response of this system.

Using (15), (14), and (10), the following solution to (31) is obtained:

$$y(t) = \frac{1}{a_2} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{a_0 + K}{a_2} \right)^m \sum_{k=0}^{m} \binom{m}{k} \left(\frac{a_1}{a_0 + K} \right)^k \times \left\{ K \mathcal{E}_m \left(t, -\frac{T_d}{a_2}; \beta_2 - \mu, \beta_2 + \mu m - \beta_1 k + 1 \right) + T_d \mathcal{E}_m \left(t, -\frac{T_d}{a_2}; \beta_2 - \mu, \beta_2 + \mu m - \beta_1 k + 1 - \mu \right) \right\}. \quad (32)$$

Using expression (32) and the gradient method for optimization, we determined the values of parameters $K = \tilde{K}$, $T_d = 3.7343$, and $\mu = 1.15$ giving almost the same response for the unit step as in the case of the design of a classic PD-controller, i.e., with stability measure $St = 2$ and damping ratio $\xi = 0.4$.

In Fig. 2, the comparison of the unit-step response of the closed loop with the fractional-order system controlled by fractional-order PD^μ-controller and the unit-step response of the closed loop with the

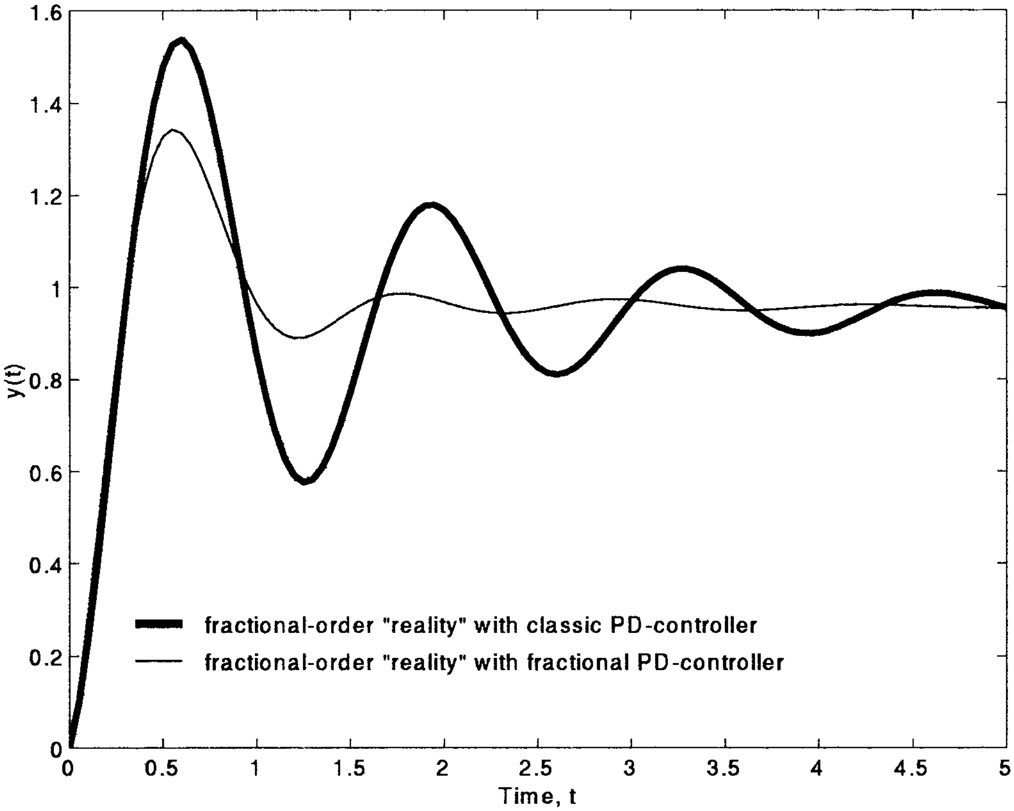

Fig. 2. Comparison of the unit-step response of the closed-loop fractional-order system with the conventional PD-controller controller, optimally designed for the approximating integer-order system (thick line), and with the PD^{μ}-controller (thin line).

same fractional-order system system controlled by the integer-order PD-controller, designed for the approximating integer-order system, is given.

This comparison shows that for satisfactory feedback control of the fractional-order system it is better to use a fractional-order controller instead of a classic integer-order controller.

IX. Concluding Remarks

We have shown that the proposed concept of the fractional-order $PI^{\lambda}D^{\mu}$-controller is a suitable way for the control of the fractional-order systems.

Of course, the physical realizations of the $PI^{\lambda}D^{\mu}$-controller circuits, which perform fractional-order differentiation and integration, are necessary. It should be mentioned that electric circuits which can serve as fractional integrators and differentiators have already been described by Oldham and Spanier [21] and by Oldham and Zoski [22].

The results of computations presented in this paper, which are based on obtained explicit solutions of the corresponding initial-value problems, were also verified by the numerical solution of those problems [10].

The most important limitation of the method presented in this paper is that only linear systems with constant coefficients can be considered. On the other hand, it allows consideration of a new class of dynamic systems (systems of an arbitrary real order) and new types of controllers.

Among the open questions, related to the questions considered in the paper, the problem of determination of the most appropriate (generally noninteger) order of the model of a real object and the problem of identification of parameters of fractional-order model must be mentioned.

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Bounds for Solutions of the Discrete Algebraic Lyapunov Equation

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***Abstract*—A family of sharp, arbitrarily tight upper and lower matrix bounds for solutions of the discrete algebraic Lyapunov are presented. The lower bounds are tighter than previously known ones. Unlike the majority of previously known upper bounds, those derived here have no restrictions on their applicability. Upper and lower bounds for individual eigenvalues and for the trace of the solution are found using the new matrix bounds. Sharp trace bounds not derivable from the matrix bounds are also presented.**

***Index Terms*—Covariance matrices, Lyapunov matrix equations, matrix bounds.**

I. Introduction

The discrete algebraic Lyapunov equation (DALE) is

$$P = A^T PA + Q, \qquad A, Q \in R^{n\times n}, \qquad Q = Q^T > 0 \tag{1}$$

where all the eigenvalues of A lie inside the unit circle, $(^T)$ and (>0) denote transpose and positive definiteness, respectively, and $P = P^T > 0$ is the solution. Bounds for solutions of the DALE are often in the form of *eigenvalue bounds*, that is bounds for single eigenvalues of P, bounds for the trace of P, or bounds for the determinant of P. A more general type of bound is a *matrix bound*, such as

$$P \le B, \qquad B = B^T \in R^{n\times n} \tag{2}$$

where the notation $P \le B$ means that the matrix $B - P$ is positive semidefinite. If one has matrix bounds, one may easily derive eigenvalue bounds.

Our particular motivation for seeking bounds for the solution of the DALE comes from the application of the Kalman filter to the problem of assimilating atmospheric data (e.g., [1]). With some simplifying assumptions, the error covariance of the estimate of the state of the atmosphere satisfies the equation in (1) with the appropriate choice of A and Q. For this application, the DALE has two distinguishing properties. First, the system comes from the discretization of a three-dimensional continuum problem; the dimension n of the matrices is large, typically of the order 10^6. Since direct treatment of (1) is impractical, estimates for the solution of the DALE are valuable and can be used, for example, to investigate the dependence of P on A and Q and to develop approximate methods. Second, in atmospheric dynamics, as in fluid dynamics, an important feature of the dynamics is nonmodal growth due to nonnormality [2], [3]. When such nonmodal growth is present, A is nonnormal and has singular values greater than one. The majority of previously known

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